18.100A MIDTERM EXAM

Thursday 22nd 2018, 9:35 - 10:50 am

1. (20 points) Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.

(1) Suppose that $\{a_n\}$ is bounded and $a_n \neq -1$. Then, $\left\{\dfrac{1}{1+a_n}\right\}$ is also bounded.

(2) Suppose that $f(x)$ is defined for $x \approx 0$, and $xf(x)$ is continuous at 0. Then, $f(x)$ is also continuous at 0.

Proof for (1). False : If $a_n = -1+\frac{1}{n}$, then $\frac{1}{1+a_n} = n \to +\infty$ as $n \to +\infty$. □

Proof for (2). False : If $f(x) = 1$ for $x \neq 0$ and $f(0) = 0$, then $xf(x) = x$ for $x \in \mathbb{R}$, which is continuous on $\mathbb{R}$. However, $f(x)$ is discontinuous at 0. □

Comments: For (2), a counter example $f(x)$ should be defined at $x = 0$. If not, $xf(x)$ is not defined at $x = 0$ and thus it is discontinuous at $x = 0$.

For example, if $f(x) = 1/x$, then $\lim_{x\to 0} xf(x) = \lim_{x\to 0} = 1$. However, the existence of a limit does not imply the continuity.

2. (25 points) Let $a_{n+1} = 2 + \sqrt{a_n}$ and $a_0 > 4$. Prove that the sequence $\{a_n\}$ is convergent, and the limit is 4.
(Hint: $x - \sqrt{x} - 2 = (\sqrt{x} - 2)(\sqrt{x} + 1)$.)

Proof. $a_0 > 4$ implies $a_1 = 2 + \sqrt{a_0} > 2 + \sqrt{4} = 4$. Assume $a_k > 4$ for some integer k, then $a_{k+1} = 2 + \sqrt{a_k} > 2 + \sqrt{4} = 4$. By the mathematical induction, we have $a_n > 4$ for all $n \in \mathbb{N}$.

Next, for each n, $a_n > 4$ implies

$$a_n - a_{n+1} = a_n - 2 - \sqrt{a_n} = (\sqrt{a_n} - 2)(\sqrt{a_n} + 1) > 0,$$

namely $a_n > a_{n+1}$ holds for every n. In conclusion, a_n is decreasing and bounded below by 4. Hence, the *completeness property* guarantees the convergence of a_n.

Let $L = \lim a_n$. Then,

$$a_n = (\sqrt{a_n})^2 = (-2 + a_{n+1})^2 = 4 - 4a_{n+1} + a_{n+1}^2.$$

Since a_{n+1} converges to L, a_{n+1}^2 converges to L^2 by the limit product rule. Thus,

$$\begin{aligned} L = \lim a_n &= \lim 4 - 4a_{n+1}^2 + a_{n+1}^2 \\ &= 4 - 4 \lim a_{n+1} + \lim a_{n+1}^2 = 4 - 4L + L^2, \end{aligned}$$

by the linearity of the limits. So, $0 = L^2 - 5L + 4 = (L - 4)(L - 1)$. The limit location theorem and $a_n > 4$ show $L \geq 4$. Hence, we have $L = 4$. □

Comments: Theorem 5.1 in the textbook does not guarantees the convergence of $\sqrt{a_n}$ for a positive convergent sequence a_n. Actually, it was the problem 2 in the second problem set. So, one should show $\lim \sqrt{a_{n+1}} = \sqrt{L}$ to use it.

Most students directly used $\lim \sqrt{a_{n+1}} = \sqrt{L}$, and lost 3 points. If one cited the problem 2 in Pset 2 without a proof, he or she lost 1 point.

3. (25 points) Suppose that S is a non-empty bounded set with $\inf S \geq 0$, and let S^2 be the set $S^2 = \{s^2 : s \in S\}$. Prove that $\sup S^2 = (\sup S)^2$.

Proof. If $\sup S = 0$, then S is the single element set $\{0\}$. So, $S^2 = \{0\}$ and $\sup S^2 = (\sup S)^2 = 0$.

We consider the other case : $\sup S > 0$.

Since S is bounded, $\sup S$ and $\inf S$ exist and $0 = \inf S \leq s \leq \sup S$ for all $s \in S$. Therefore, we have $0 \leq s^2 \leq (\sup S)^2$. Hence, S^2 is bounded and

$$\sup S^2 \leq (\sup S)^2.$$

Since $\sup S > 0$, there exists an element $0 < s_1 \in S$. Then, for any $s \in S$ with $s_1 \leq s$, the definition $s^2 \leq \sup S^2$ yields

$$s \leq \frac{\sup S^2}{s} \leq \frac{\sup S^2}{s_1}. \tag{1}$$

Therefore, we have

$$s \leq \frac{\sup S^2}{s_1}$$

for all $s \in S$, namely $\sup S \leq \sup S^2/s_1$. Since $\sup S > 0$, we have

$$s_1 \leq \frac{\sup S^2}{\sup S},$$

for all $0 < s_1 \in S$. Hence, $\sup S \leq \sup S^2/\sup S$, namely $(\sup S)^2 \leq \sup S^2$.

In conclusion, $\sup S^2 = (\sup S)^2$.

□

Comments: So many students deduced directly from $s \leq \sup S^2/s$ that $\sup S^2/s$ is an upper bound. However, $\sup S^2/s$ varies as s varies. However, an upper bound is a fixed number. So, it is necessary to consider a fixed value $\sup S^2/s_1$ as (1).

4. (10 points) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} 2^n n^2 x^{2n}$, and explain why.

Proof. Set $a_n = 2^n n^2 x^{2n}$. Then, if $x \neq 0$

$$\left|\frac{a_{n+1}}{a_n}\right| = 2x^2\left(1 + \frac{1}{n}\right)^2.$$

By Theorem 5.1 and $\lim \frac{1}{n} = 0$, we have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = 2x^2.$$

Hence, by the ratio test, the power series is convergent if $2x^2 < 1$ and divergent if $2x^2 > 1$. (It is inconclusive if $2x^2 = 1$.) Namely, it is convergent if $|x| < 1/\sqrt{2}$ and divergent if $|x| > 1/\sqrt{2}$. So, $1/\sqrt{2}$ is the radius of convergence.

□

Comments: One should mention that it diverges if $|x| > 1/\sqrt{2}$.

5. (20 points) Let $f(x)$ be an increasing function defined for $x \in (-\infty, +\infty)$. Suppose that given any two rational numbers $r, q \in \mathbb{Q}$, we have

$$|f(r) - f(q)| \leq |r - q|.$$

Prove that $f(x)$ is continuous on $\mathbb{R}$.

(You may need to use that fact that given any two different real numbers $x < y$, there exists a rational number r such that $x < r < y$.)

Proof. Given $x_0 \in \mathbb{R}$, there exist two sequences of rational numbers $a_n, b_n \in \mathbb{Q}$ with $n \in \mathbb{N}$ such that $x_0 - \frac{1}{n} < a_n < x_0$ and $x_0 < b_n < x_0 + \frac{1}{n}$.

Since $f(x)$ is increasing, for $x \in (a_n, b_n)$ the following holds

$$|f(x) - f(x_0)| \leq |f(a_n) - f(b_n)|.$$

Moreover, by definition of a_n, b_n we have

$$|f(a_n) - f(b_n)| = |a_n - b_n| \leq \left|(x_0 - \frac{1}{n}) - (x_0 + \frac{1}{n})\right| = \frac{2}{n}.$$

In conclusion, given $\epsilon > 0$, we have $|f(x) - f(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$ where $\delta = \min\{|x_0 - a_N|, |x_0 - b_N|\}$ for some $N > 2/\epsilon$. Namely, f is continuous at any $x_0 \in \mathbb{R}$. So, $f(x)$ is continuous on $\mathbb{R}$.

□

Comments: In the last line of the proof above, $(x_0 - \delta, x_0 + \delta) \subset (a_N, b_N)$ by $\delta = \min\{|x_0 - a_N|, |x_0 - b_N|\}$. Hence, in the δ-neighborhood, we have $|f(x) - f(x_0)| \leq |f(a_N) - f(b_N)| \leq 2/N < \epsilon$.

6.(10 points, bonus problem) Let $f(x)$ be defined for $x \in (-\infty, +\infty)$. Suppose that given any two real numbers $x, y \in \mathbb{R}$

$$(*) \qquad tf(x) + (1-t)f(y) \geq f(tx + (1-t)y),$$

holds for all $t \in [0,1]$. Prove that $f(x)$ is continuous on $\mathbb{R}$.
(We say $f(x)$ is a convex function if $(*)$ holds.)

Proof. Given $x_0 \in \mathbb{R}$, we define

$$\begin{aligned} g(x) =& f(x_0) + \big(f(x_0+1) - f(x_0)\big)(x - x_0), \\ h(x) =& f(x_0) + \big(f(x_0) - f(x_0-1)\big)(x - x_0). \end{aligned}$$

Then, for $t \in [0,1]$

$$g(x_0+t) = tf(x_0+1) + (1-t)f(x_0) \geq f(x_0+t).$$

Also, we have

$$\frac{1}{1+t}f(x_0+t) + \frac{t}{1+t}f(x_0-1) \geq f\Big(\frac{x_0+t}{1+t} + \frac{tx_0-t}{1+t}\Big) = f(x_0),$$

and thus

$$h(x_0+t) = (1+t)f(x_0) - tf(x_0-1) \leq f(x_0+t).$$

Since $\lim\limits_{t\to 0} g(x_0+t) = \lim\limits_{t\to 0} h(x_0+t) = f(x_0)$, the squeeze theorem shows that $\lim\limits_{x\to x_0^+} f(x) = f(x_0)$.

In the same manner, we can show $g(x_0-t) \leq f(x_0-t) \leq h(x_0-t)$ for $t \in [0,1]$. So, the squeeze theorem shows $\lim\limits_{x\to x_0^-} f(x) = f(x_0)$. Therefore, $f(x)$ is continuous at any $x_0 \in \mathbb{R}$. Hence, $f(x)$ is continuous on $\mathbb{R}$. $\square$

Comments: To understand the idea of the proof above, draw the graphs of f, g, h. The given condition means that if the graph of $f(x)$ intersects with a line at two point $(a, f(a))$ and $(b, f(b))$, then $f(x)$ is less than or equal to the y-value of the line on the interval $[a,b]$. However, $f(x)$ is grater than or equal to the y-value of the line out of the interval $[a,b]$.

Now, the graphs of g, h are two lines cross at $(x_0, f(x_0))$. Hence, one can use the squeeze theorem by using the two lines.